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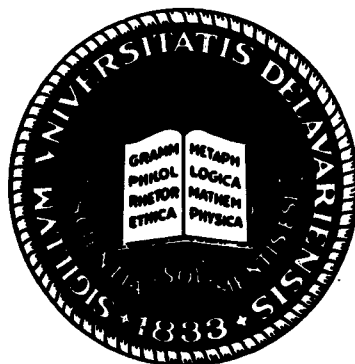
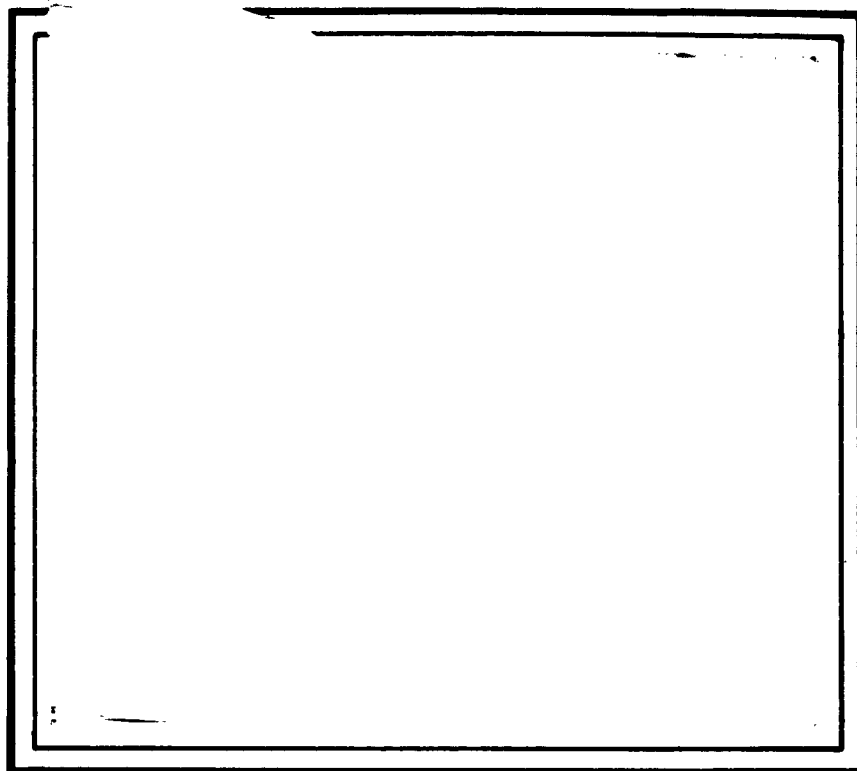


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QUASI-INTERIOR POINTS OF CONES

by

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Abstract

A point $x \neq \theta$ belonging to a convex cone K with vertex θ in a locally convex linear topological space X is called a quasi-interior point (QI-point) of K if the linear extension of the set $K \cap (x - K)$ is dense in X . The set K_q of all quasi-interior points of K is called the quasi-interior of K . Many properties of QI-points and of cones with non-void quasi-interiors are determined. Among the results established are the following.

If K has a non-void interior K° then $K_q = K^\circ$. Examples are given to show that a cone with void interior may have a non-void quasi-interior.

Let K and K' be cones with non-void quasi-interiors K_q and K'_q such that $K_q \cap K'_q = \emptyset$. If H is a hyperplane separating K and K' then H strictly separates K_q and K'_q .

Each QI-point of K is a non-support point of K .

If $K_q \neq \emptyset$ and C is a convex set with non-void interior C° such that $C^\circ \cap K_q = \emptyset$ then there exists a hyperplane H separating C and K and strictly separating C° and K_q .

If $K \cap (-K) = \{\theta\}$, $K_q \neq \emptyset$ and $x \in K$, $x \notin K_q$, $x \neq \theta$ there exists a subset H of X maximal with respect to the properties: (i) H is a proper linear subspace of X , (ii) $x_0 \in H$, (iii) $H \cap K_q = \emptyset$. Furthermore, if $S_1 = H + K_q$ and $S_2 = H - K_q$ we have $H \cap S_1 = H \cap S_2 = S_1 \cap S_2 = \emptyset$; $H \cup S_1 \cup S_2 = X$; $K \subset H \cup S_1$; $K_q \subset S_1$; and $x_1 \in S_1$ and $y \in H$ imply an $x_2 \in S_2$ such that $y = \frac{1}{2}(x_1 + x_2)$. It is conjectured that H is necessarily either a hyperplane or is dense in X .

1. Introduction.

In May 1963 Professor R. E. Fullerton died at the age of 47 as the result of injuries received in an automobile accident while on a lecture tour of Europe. A mathematical inventory of his effects produced the beginnings of several papers. Although the notes found were, for the most part, incomplete and fragmentary, it was decided that an attempt should be made to complete and publish the work so tragically interrupted. The writer, being a former student of Fullerton and having maintained close professional contact with him, was selected to make this attempt. The paper which will result from this report will be the first of several to be written in Fullerton's specialty, namely, the application of geometrical techniques to Functional Analysis.

The notion of a quasi-interior point of a cone in a linear topological space was devised by Fullerton in a research report [4]. His purpose in introducing this concept and discussing several properties possessed by cones with quasi-interior points was to lay a foundation upon which a realistic generalization of positive operator theory could be based. Of particular interest were the results of Krein and Rutman [8] concerning the spectral theory of linear operators leaving invariant a cone in a Banach space. These results required that the positive cone have a non-void interior and therefore were not applicable in many interesting function

spaces. (For example, the positive cone in an infinite dimensional L-space has no interior points. c.f. [2].) An actual application of the quasi-interior point concept to extending some of the Krein and Rutman results was subsequently carried out by H. Schaefer [9] who arrived at the technique independently.

The research interest of this writer has, in the past few years, been centered in the theory of infinite programming in linear spaces. His attention was recently once again focused upon the quasi-interior point concept when, in the course of attempting to extend certain results in the use of Lagrangian saddle-points for non-linear programming, a need was found for results concerning separation and support properties of cones with void interiors. The principal results in this report involve such properties of cones with non-void quasi-interiors but having no interior points. Fullerton's original investigations concerning quasi-interior points are included in this report for the sake of completeness and because the report [4] in which they first appeared was never published and is now no longer accessible. A few of these basic results can be found also in [10].

The writer feels obliged to make a comment concerning the role of technical reports in general and this report in particular. He believes that a technical report serves at least two very useful purposes. The first is in its role as a pre-print of a future publication, making available to the

interested reader material which will ordinarily not appear in published form for many months and yet, in a format to which reference can be made by workers wishing to make use of the results. The second is the possibility of including many details and insights which, of necessity, must be omitted from the published version, yet which can be extremely helpful to the reader.

2. Basic definitions and notation.

Included in this section are the definitions and notation which will be used throughout the remainder of the report without further reference.

The underlying space in all of the discussions will be a locally convex real linear topological space, always denoted by X . That is, X is a real linear space with elements x, y, z, \dots and scalars $\alpha, \beta, \gamma, \dots$ belonging to the real number system R , together with a topology in which the operations of addition (+) and scalar multiplication (\cdot) are continuous from $X \times X$ to X and $R \times X$ to X , respectively. Furthermore, this topology is such that the family of convex neighborhoods of the zero element (denoted by θ to distinguish it from the scalar 0 and the empty set \emptyset) forms a local base - i.e. each neighborhood of θ contains a convex neighborhood of θ . R itself is always assumed to possess the usual topology.

It will be clear to the reader that many of the results to follow will be valid in linear spaces with considerably

weaker topological properties. Indeed, many of the results, being purely algebraic, require no topology at all. The reason for requiring that X be a locally convex space is one of economy and the proofs themselves will indicate to what extent the topological conditions can be relaxed.

The line segment joining points x and y in X is denoted by $[x, y]$ and defined by $[x, y] = \{z : z = \alpha x + (1 - \alpha)y, 0 \leq \alpha \leq 1\}$. We shall also use the notations $(x, y) = \{z : z = \alpha x + (1 - \alpha)y, 0 < \alpha < 1\}$, $[x, y) = \{z : z = \alpha x + (1 - \alpha)y, 0 < \alpha \leq 1\}$, and $(x, y] = \{z : z = \alpha x + (1 - \alpha)y, 0 \leq \alpha < 1\}$. The line in X determined by points x and y is denoted by $L(y, x)$ and defined by $L(y, x) = \{z : z = \alpha x + (1 - \alpha)y, \alpha \in \mathbb{R}\}$. The ray, or half-line from y through x is $R(y, x) = \{z : z = \alpha x + (1 - \alpha)y, \alpha \geq 0\}$.

A set C in X is convex if for any pair of elements x, y in C it follows that $[x, y] \subset C$. The symbol K will always be used to denote a convex cone with vertex θ in X . That is, K is a subset of X such that $K + K \subset K$ and $\alpha K \subset K$ for all $\alpha \geq 0$. (By $A + B$ we mean $\{x + y : x \in A, y \in B\}$ and by αA we mean $\{\alpha x : x \in A\}$.)

For a subset A of X , $[A]$ denotes the linear extension of A and is the set of all finite linear combinations of elements in A . As is well known, $[A]$ is the smallest linear subspace containing A . It is easily seen that if K is a convex cone in X then the statements $K - K = X$ and $[K] = X$ are equivalent. A subset A will be said to generate X if $[A]$ is

dense in X -i.e. if X is the closure of $[A]$.

A variety is a translate of a linear subspace. By a hyperplane we mean a closed maximal proper linear subspace of X . That is, H is a hyperplane if H is a linear subspace of X , H is properly contained in no proper linear subspace of X , and H is closed. It is well known that H is a hyperplane in X if and only if, for some $x \in X$, $x \notin H$, $X = H + Rx$.

A° and \bar{A} will, as is customary, denote respectively, the interior and closure of the set A .

The notation $A \sim B$ is used to denote $\{x : x \in A, x \notin B\}$ and is to be distinguished from the notation $A - B = \{x - y : x \in A, y \in B\}$.

3. Quasi-interior points of cones. Basic properties.

3.1 Definition. A point $x \in K$, $x \neq \theta$ is called a quasi-interior point (QI-point) of K if the set $P_x = K \cap (x-K)$ generates X . The set of all QI-points of K will be denoted by K_q and is called the quasi-interior of K . If $x \in K$ and $x \notin K_q$, x is called a quasi-boundary point of K . The set of all quasi-boundary points of K is called the quasi-boundary of K and is denoted by K_b . That is $K_b = K \sim K_q$.

3.2 Lemma. (a) $\lambda > 0 \Rightarrow P_{\lambda x} = \lambda P_x$.

(b) $u \in P_x \Rightarrow P_u \subset P_x$.

(c) $y \in x + K \Rightarrow P_x \subset P_y$.

Proof: Suppose $y \in P_x$ and $\lambda > 0$. Then $y \in K$ hence $\lambda y \in K$. Also $y \in x - K$ so $y = x - v$ where $v \in K$ hence

$\lambda y = \lambda x - \lambda v \in \lambda x - K$. Thus $\lambda y \in P_{\lambda x}$ and $\lambda P_x \subset P_{\lambda x}$.
 Conversely, suppose $y \in P_{\lambda x}$, $\lambda > 0$. Then $y \in K$ and
 $y \in \lambda x - K$ so $y = \lambda x - w$ with $w \in K$. It follows that
 $y = \lambda(x - \frac{1}{\lambda}w)$ hence $\frac{1}{\lambda}y = x - \frac{1}{\lambda}w \in x - K$ and $\frac{1}{\lambda}y \in P_x$ so
 $y \in \lambda P_x$. Thus $P_{\lambda x} \subset \lambda P_x$ and (a) has been proved. If $u \in P_x$
 then $u \in K$ and $u = x - v_1$ with $v_1 \in K$. Suppose $w \in P_u$. Then
 $w \in K$ and $w = u - v_2$ with $v_2 \in K$. Consequently $w =$
 $(x - v_1) - v_2 = x - (v_1 + v_2) \in x - K$ so $P_u \subset P_x$ and (b)
 holds. Finally, to prove (c) suppose $y \in x + K$. Then
 $y = x + u$ where $u \in K$. If $w \in P_x$ then $w \in K$ and $w = x - v$
 where $v \in K$. But then $y = x + u = (w + v) + u = w + (v + u)$
 so $w = y - (v + u) \in K \cap (y - K) = P_y$. Thus $P_x \subset P_y$. The
 lemma has been proved.

3.3 Lemma. If C is a non-trivial convex set containing
 θ then each element of $[C]$ lies on a line determined by two
 points of C .

Proof: Suppose $y \in [C]$. Then $y = \sum_{i=1}^n \alpha_i x_i$ where for
 $i = 1, 2, \dots, n$, $x_i \in C$ and $\alpha_i \in \mathbb{R}$. If all the α_i are
 zero then $y = \theta \in C$ and for any $x \in C$, $x \neq \theta$, y lies on the
 line $L(\theta, x)$. Assume therefore that $\alpha_i \neq 0$ for $i = 1, 2, \dots, n$.
 If the α_i are all positive, let $\lambda = \sum_{i=1}^n \alpha_i > 0$ and let
 $y_1 = \frac{1}{\lambda} \sum_{i=1}^n \alpha_i x_i$. Since C is convex, $y_1 \in C$ and $y = \lambda y_1$ lies
 on the line $L(\theta, y_1)$. If all the α_i are negative, let $\lambda =$
 $\sum_{i=1}^n (-\alpha_i) > 0$ and $y_1 = \frac{1}{\lambda} \sum_{i=1}^n (-\alpha_i) x_i$. Then $y_1 \in C$ and
 $y = -\lambda y_1$ again lies on $L(\theta, y_1)$. Finally assume that some

of the α_i are positive and some are negative and separate the positive and negative coefficients. That is, write $y = \sum_{j=1}^m \alpha_{1j} x_{1j} - \sum_{k=1}^r (-\alpha_{1k}) x_{1k}$ where $\alpha_{1j} > 0$ for $j = 1, 2, \dots, m$ and $\alpha_{1k} < 0$ for $k = 1, 2, \dots, r$ with $n = m + r$. Let $\lambda_1 = \sum_{j=1}^m \alpha_{1j} > 0$ and $\lambda_2 = \sum_{k=1}^r (-\alpha_{1k}) > 0$. Setting $y_1 = \frac{1}{\lambda_1} \sum_{j=1}^m \alpha_{1j} x_{1j}$ and $y_2 = \frac{1}{\lambda_2} \sum_{k=1}^r (-\alpha_{1k}) x_{1k}$ we have $y_1 \in C$, $y_2 \in C$ and $y = \lambda_1 y_1 - \lambda_2 y_2$. If $\lambda_1 = 1$ then, since $\lambda_2 > 0$, $\frac{1}{1 + \lambda_2} \leq 1$ so, setting $y_1' = \frac{1}{1 + \lambda_2} y_1$ we have $y_1' = \frac{1}{1 + \lambda_2} y_1 + \frac{\lambda_2}{1 + \lambda_2} \theta \in C$ and $y = (1 + \lambda_2) y_1' - \lambda_2 y_2$ whence y lies on the line $L(y_1', y_2)$ determined by the points y_1' , y_2 belonging to C . If $\lambda_1 \neq 1$ let $y_2' = \frac{-\lambda_2}{1 - \lambda_1} y_2$ and $y_1' = \frac{\lambda_1}{1 + \lambda_2} y_1$. Then $y = \lambda_1 y_1 + (1 - \lambda_1) y_2' = (1 + \lambda_2) y_1' - \lambda_2 y_2$. If $\frac{\lambda_1}{1 + \lambda_2} \geq 1$ then $\lambda_1 \geq 1 + \lambda_2$ and $\lambda_1 - 1 \geq \lambda_2 > 0$ so $\frac{\lambda_2}{\lambda_1 - 1} \leq 1$ and $y_2' = \frac{\lambda_2}{\lambda_1 - 1} y_2 \in C$. Then $y = \lambda_1 y_1 + (1 - \lambda_1) y_2'$ lies on the line $L(y_1, y_2')$ determined by two points of C . If $\frac{\lambda_1}{1 + \lambda_2} < 1$ then $y_1' = \frac{\lambda_1}{1 + \lambda_2} y_1 \in C$ so $y = (1 + \lambda_2) y_1' - \lambda_2 y_2$ lies on $L(y_1', y_2)$ determined by two points of C . In every case, therefore, the lemma's conclusion has been verified.

3.4 Lemma. If $y \in [P_x]$ then there exist points u and v in $P_x \cap L(y, \frac{x}{2})$ such that $x = u + v$. That is, $\frac{x}{2}$ is the mid-point of a segment $[u, v] \subset P_x \cap L(y, \frac{x}{2})$.

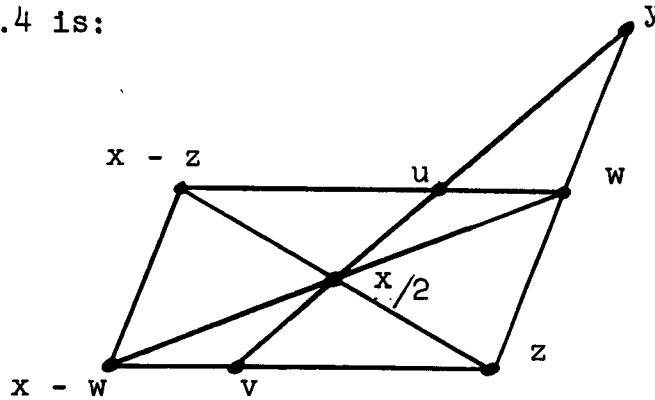
Proof: Clearly P_x is convex being the intersection of the convex sets K and $x - K$. Thus, by Lemma 3.3, if $y \in [P_x]$, y lies on $L(w, z)$ with $w, z, \in P_x$. From the definition of P_x clearly $x - w$ and $x - z$ also belong to P_x . Suppose first

that $y \in [w, z]$. Then, because P_x is convex, $y \in P_x$ and $\frac{x}{2}$ is the midpoint of the segment $[u, v]$ where $u = x \in P_x$ and $v = x - y \in P_x$. Now suppose that $y \notin [w, z]$. Then, by interchanging w and z if necessary, $y = \gamma w + (1 - \gamma)z$ for some real scalar $\gamma < 0$. By an elementary calculation,

$$y = \gamma \left\{ \frac{1 - \gamma}{1 - 2\gamma} (x - z) + \frac{-\gamma}{1 - 2\gamma} w \right\} + (1 - \gamma) \left\{ \frac{-\gamma}{1 - 2\gamma} (x - w) + \frac{1 - \gamma}{1 - 2\gamma} z \right\}.$$

Setting $u = \frac{1 - \gamma}{1 - 2\gamma} (x - z) + \frac{-\gamma}{1 - 2\gamma} w$ and $v = \frac{-\gamma}{1 - 2\gamma} (x - w) + \frac{1 - \gamma}{1 - 2\gamma} z$ we see that u and v belong to P_x , $y \in L(u, v)$ and $\frac{x}{2}$ is the midpoint of $[u, v]$.

The pictorial description of the situation described in Lemma 3.4 is:



3.5 Definition. If M is a subset of X a point x in M is called a directionally-interior point (DI-point), (or internal point, radial point, core point) of M if for each $y \in M$, $y \neq x$ there is a $z \neq x$ such that $[x, z] \subset [x, y] \cap M$. Equivalently, x is directionally-interior point of M if, given any line L in X through x there exist points y and z in $L \cap M$ such that $x \in (y, z) \subset M$. The set of all directionally

interior points of M is called the directional-interior (also radial kernel or core) of M and will be denoted by M_d .

3.6 Corollary. $[P_x] = X$ if and only if $\frac{x}{2}$ is directionally-interior to P_x .

Proof: This follows immediately from Definition 3.5 and Lemma 3.4.

3.7 Theorem. (a) If $x \in K^0$ then $[P_x] = X$.

(b) If K is closed, X is of the second category and $[P_x] = X$, then $x \in K^0$.

Proof: Suppose $x \in K^0$. Then $\frac{x}{2} \in K^0$ and $\frac{x}{2} \in (x - K)^0$. Let N' be an open set such that $\frac{x}{2} \in N' \subset K$ and let N'' be an open set such that $\frac{x}{2} \in N'' \subset x - K$. Then $N = N' \cap N''$ is an open set such that $\frac{x}{2} \in N \subset P_x$. It is a well known result that if P_x contains an open set, $[P_x] = X$. Thus, (a) is proved. Suppose $[P_x] = X$. By Corollary 3.6, $\frac{x}{2}$ is directionally interior to P_x . If K is closed so is P_x and trivially P_x is a convex Baire set. By a theorem of Klee [6], if X is of the second category, the directional-interior of P_x is a subset of its interior. Since $\frac{x}{2}$ is interior to P_x , clearly, $\frac{x}{2} \in K^0$.

3.8 Example. The following example shows that the category assumption in Theorem 3.7 (b) is necessary. The details of the verification may be found in [3].

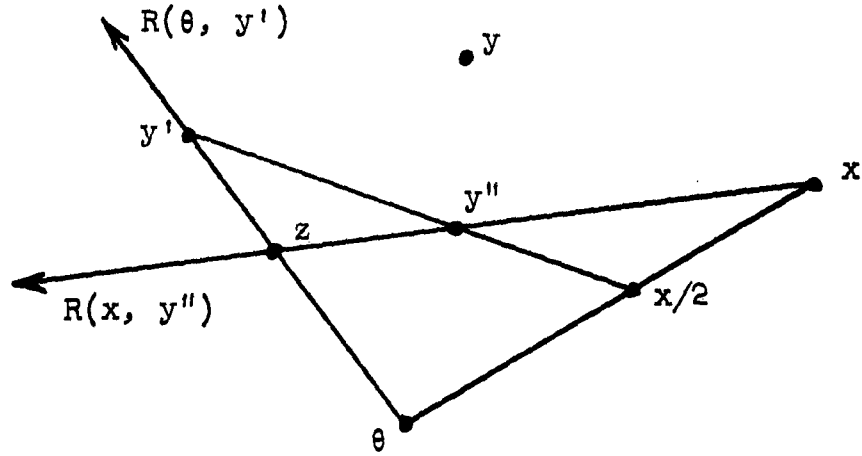
Let S denote the real linear space whose elements are

all real sequences with at most a finite number of non-zero coordinates. Let x_0 denote the sequence $\{\frac{1}{2^i}\}_{i=1}^{\infty}$. The space X is defined by the equation $X = S + Rx_0$. If $x = \{\xi_i\}$ is any element of X define $\|x\| = \sum_{i=1}^{\infty} |\xi_i|$. Let K be the set of all points $x = \{\xi_i\}$ in X such that, for all i , $\xi_i \geq 0$. Then X is a normed linear space of the first category in itself, K is a closed convex cone in X , $K^0 = \emptyset$ and yet $[P_{x_0}] = X$.

3.9 Theorem. (a) $K^0 \subset K_q$.

(b) If $K^0 \neq \emptyset$ then $K_q \subset K^0$.

Proof: If $K^0 = \emptyset$ trivially $K^0 \subset K_q$. If $K^0 \neq \emptyset$ and $x \in K^0$, by Theorem 3.7 (a), $[P_x] = X$ so, clearly, P_x generates X and $x \in K_q$. Thus, (a) has been proved. To prove (b) suppose that x is any element of K_q and y is any element of K^0 . Since $y \in K^0$ there is an open set N_y such that $y \in N_y \subset K$. Since $x \in K_q$, $[P_x]$ is dense in X and there exists a point $y' \in [P_x] \cap N_y$. Let N'_y be an open set such that $y' \in N'_y \subset N_y$. By Lemma 3.4 the line $L(y', \frac{x}{2})$ contains a point y'' in P_x such that $y'' \in (\frac{x}{2}, y')$, say $y'' = \gamma y' + (1-\gamma)\frac{x}{2}$ with $0 < \gamma < 1$. If y'' is a multiple of x , so is y' and hence, since $y' \in K^0$, so is x . If y'' is not a multiple of x then θ , x and y'' determine a two dimensional subspace E of X . Consider the rays $R(\theta, y') = \{\alpha y' : \alpha \geq 0\}$ and $R(x, y'') = \{\beta x + (1 - \beta)y'' : \beta \leq 1\}$.



The rays $R(\theta, y')$ and $R(x, y'')$ intersect in the point z where, since $z = \alpha y' = \beta x + (1 - \beta)y''$ and $y'' = \gamma y' + (1 - \gamma)\frac{x}{2}$ we have $\alpha y' = \beta x + (1 - \alpha)(\gamma y' + (1 - \gamma)\frac{x}{2}) = [\beta + \frac{(1-\alpha)(1-\gamma)}{2}]x + (1 - \beta)\gamma y'$. Since x and y' are independent, $\beta + \frac{(1-\alpha)(1-\gamma)}{2} = 0$ and $\alpha = (1 - \beta)\gamma$. Solving for β gives $\beta = \frac{\gamma - 1}{\gamma + 1}$ and $\alpha = \frac{2\gamma}{\gamma + 1}$. Thus $z = \frac{2\gamma}{\gamma + 1}y'$. The open set $N_z = \frac{2\gamma}{\gamma + 1}N_{y'}$ contains z and is contained in K and, hence, z is interior to K . Also, since $R(\theta, y') \subset K$ and $R(x, y'') \subset x - K$ (to see this note that $x - R(x, y'') = \{x - [\beta x + (1 - \beta)y''] : \beta \leq 1\} = \{(1 - \beta)(x + y'') : \beta \leq 1\} \subset K$) we have $z \in K \cap (x - K) = P_x$. Since P_x is symmetric with respect to $\frac{x}{2}$, $x - z \in P_x$ so $x - z \in K$ and, in particular, $x - z + N_z \subset K$. The set $\frac{1}{2}\{(x - z) + N_z\}$ is an open set contained in K and contains $\frac{x}{2}$. (To see that this set contains $\frac{x}{2}$ we note that $z \in N_z$ so $\frac{1}{2}z \in \frac{1}{2}N_z$ and $\frac{x}{2} \in \frac{1}{2}x - \frac{1}{2}z + \frac{1}{2}N_z$. Thus, $\frac{x}{2}$ is interior to K and, consequently, $x \in K^\circ$.

3.10 Corollary. If $K^\circ \neq \emptyset$ then $K^\circ = K_q$.

3.11 Corollary. If $K^0 \neq \emptyset$ and $x \in K_q$ then P_x has a non-void interior.

3.12 Theorem. A necessary and sufficient condition that $x \in K_q$ is that for any $y \in X$ and any open set N containing y there exists a line L with $L \cap N \neq \emptyset$ such that x is interior to the segment $L \cap K$ relative to L .

Proof: Let z be any point of X and let N_z be any open set containing z . Let $y = 2z$ and $N_y = 2N_z$. There exists a point $y' \in N_y$ such that the line $L(y', x)$ is such that $x \in (u, v)$ where $[u, v] \subset L(y', x) \cap K$. Then $\frac{1}{2}y' \in N_y$ and $L(\frac{y'}{2}, \frac{x}{2}) = \frac{1}{2}L(y', x)$. $\frac{x}{2} \in (\frac{u}{2}, \frac{v}{2})$ and $[\frac{u}{2}, \frac{v}{2}] \subset L(\frac{y'}{2}, \frac{x}{2}) \cap K$. Thus, if x has the above property, so does $\frac{x}{2}$. If $\frac{x}{2}$ is interior to $L \cap K$ then, by the symmetry of P_x with respect to $\frac{x}{2}$, $\frac{x}{2}$ is interior to $L \cap (x - K)$ and hence interior to $L \cap P_x$. If $y \in X$, $y \in N_y$ and if there exists a line L with $\frac{x}{2}$ interior to $L \cap P_x$ and with $L \cap N_y \neq \emptyset$, $[P_x]$ is dense in X and x is a QI-point.

Conversely, let x be any QI-point of K . Assume that for some $y \in X$ there exists a neighborhood N_y such that any line L through $\frac{x}{2}$ with $L \cap N_y \neq \emptyset$ does not have $\frac{x}{2}$ interior to $L \cap K$. By symmetry we have $L \cap P_x = \{\frac{x}{2}\}$ for all such lines L . Since $x \in K_q$ there must exist a point $z \in N_y$ with $z \in [P_x]$. By Lemma 3.4, z lies on a line L' through $\frac{x}{2}$ such that $L' \cap P_x$ contains $\frac{x}{2}$ as an interior point of the segment. This contradiction completes the proof.

3.13 Theorem. If K is closed and X is of the second category then $x \in K_q$, $x \notin K^0$ imply $X \sim [P_x]$ is dense in X .

Proof: Suppose $X \sim [P_x]$ is not dense in X . Then there exists a $y \in X$ and an open set N such that $y \in N \subset [P_x]$. This implies, that $[P_x] = X$ and hence, by Theorem 3.7 (b), $x \in K^0$, a contradiction. Thus, $X \sim [P_x]$ is dense in X and $[P_x]$ and $X \sim [P_x]$ are complementary dense subsets of X .

3.14 Theorem. (a) K_q is a convex subset of K .

(b) $K_q + K \subset K_q$.

(c) $K_q = \bigcup \{x + K : x \in K_q\}$.

(d) $K_q \cup \{0\}$ is a convex cone.

(e) $K - K = X$ and $K_q \neq \emptyset$ imply

$$K_q - K_q = X.$$

Proof: If $K_q = \emptyset$, (a) holds trivially. Suppose $x, y \in K_q$ and let $z = \alpha x + (1 - \alpha)y$, $0 < \alpha < 1$. It may be assumed without loss of generality that $\alpha \leq \frac{1}{2}$. Since $z - \alpha x = (1 - \alpha)y$ and therefore $z - \alpha x = u \in K$ we have $\alpha x = z - u \in z - K$ and, since $\frac{1}{\alpha}K = K$ it follows that $x \in \frac{1}{\alpha}z - K$. Also setting $v = \alpha x + (1 - \alpha)y - \alpha y = \alpha x + (1 - 2\alpha)y$ then $\alpha > 0$, $1 - 2\alpha \geq 0$ and, since $x, y \in K$ so is $v \in K$. Thus, $\alpha y = z - v \in z - K$ and $y \in \frac{1}{\alpha}z - K$. This implies that x and y belong to $K \cap (\frac{1}{\alpha}z - K) = P_{z/\alpha}$. By Lemma 3.2 (b), $x \in P_{z/\alpha}$ implies $P_x \subset P_{z/\alpha}$ and since $[P_x]$ is dense in X , so is $[P_{z/\alpha}]$. By Lemma 3.2 (a), $P_{z/\alpha} = \frac{1}{\alpha}P_z$ so $[P_z] = \alpha[\frac{1}{\alpha}P_z] = \alpha[P_{z/\alpha}]$ is dense in X and $z \in K_q$. Thus K_q is convex and (a) is verified.

If $y \in x + K$ then $P_x \subset P_y$, by Lemma 3.2, so if $[P_x]$ is dense in X so is $[P_y]$. Thus, (b) holds.

(c) follows immediately from (b).

To prove (d) we note that, from (b) it follows trivially that $K_q + K_q \subset K_q$. That $\alpha K_q \subset K_q$ for $\alpha > 0$ follows from Lemma 3.2 (a). Thus $K_q \cup \{0\}$ is a convex cone.

Finally, suppose $K - K = X$. Then, for any $x \in X$, $x = y - z$ with $y, z \in K$. Let $u \in K_q$. Then, by part (b), $y + u, z + u \in K_q$ so $x = (y + u) - (z + u) \in K_q - K_q$, and (e) is proved.

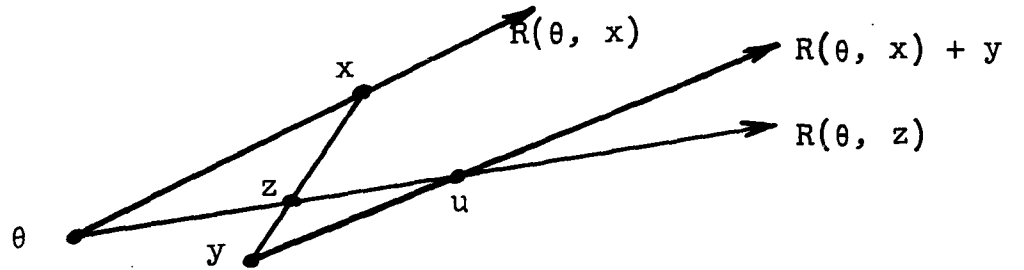
3.15 Theorem. If $[u, v] \subset K$ and $(u, v) \cap K_q \neq \emptyset$ then $(u, v) \subset K_q$.

Proof: Suppose $z \in (u, v) \cap K_q$ and let x be any point in (u, v) . There exists a $y \in (u, v)$ and a real number α , $0 < \alpha < 1$ such that $x = \alpha z + (1 - \alpha)y$. $\alpha z = x - (1 - \alpha)y \in x - K$ so $z \in (\frac{1}{\alpha}x - K) \cap K = P_{x/\alpha}$. By Lemma 3.2 (b), $P_z \subset P_{x/\alpha}$ and, since $[P_z]$ is dense in X so is $[P_{x/\alpha}]$ and, hence, $[P_x]$ is dense in X . Thus $x \in K_q$. Since x was any point in (u, v) , $(u, v) \subset K_q$.

3.16 Theorem. If $K_q \neq \emptyset$ then $K \subset \overline{K_q}$.

Proof: Suppose that $K_q \neq \emptyset$ but that $K \not\subset \overline{K_q}$. Then there exists an $x \in K$ such that $x \notin \overline{K_q}$ and, hence, there exists an open set N containing x such that $N \cap K_q = \emptyset$. Let y be any element in K_q and let $U = N - x$. U is an open set containing θ and there is an open set V such that $\theta \in V \subset U$ with V

absorbing (i.e. V is such that to each $z \in X$ corresponds an $\epsilon > 0$ with $\beta z \in V$ if $0 < |\beta| \leq \epsilon$). In particular, if $v = y - x$ there exists a β_0 , $0 < \beta_0 < 1$ such that $\beta_0 v = \beta_0 y - \beta_0 x \in V \subset U$ so $\beta_0 y - \beta_0 x \in N - x$ or $\beta_0 y + (1 - \beta_0)x \in N$. Let $z = \beta_0 y + (1 - \beta_0)x$. Then $z \in N \cap (x, y)$. Since $N \cap K_q = \emptyset$, $z \notin K_q$. Consider the rays $R(\theta, x)$ and $R(\theta, z)$. Now $R(\theta, x) + y = \{y + \lambda x : \lambda \geq 0\}$ and $R(\theta, z) = \{\lambda(\beta_0 y + (1 - \beta_0)x) : \lambda \geq 0\}$. It is easily seen that the element $u = y + \frac{(1 - \beta_0)}{\beta_0}x = \frac{1}{\beta_0}(\beta_0 y + (1 - \beta_0)x)$ is the point of intersection of the rays $R(\theta, x) + y$ and $R(\theta, z)$.



Clearly $u - y = \frac{(1 - \beta_0)}{\beta_0}x \in K$ so $y \in K \cap (u - K) = P_u$. By Lemma 3.2 (b), $P_y \subset P_u$ so, since $y \in K_q$, $[P_u]$ is dense in X and $u \in K_q$. But $z = \beta_0 u$ and $\beta_0 > 0$ so, by Theorem 3.14 (d), $z \in K_q$. This contradiction completes the proof.

3.17 Corollary. If K is closed and $K_q \neq \emptyset$ then $K = \overline{K_q}$.

3.18 Theorem. If $x \in K_q$ and $y \in K_p$ then $[x, y) \subset K_q$.

Proof: Suppose $z \in (x, y)$. Then $z = \beta x + (1 - \beta)y$ with $0 < \beta < 1$. As in the proof of Theorem 3.16, the point $u = x + \frac{(1 - \beta)}{\beta}y = \frac{1}{\beta}(\beta x + (1 - \beta)y)$ is common to $R(\theta, y) + x$

and $R(\theta, z)$ and it follows as before, that $z \in K_Q$.

3.19 Theorem. Let $x \in K_Q$, $x \notin K^0$. If $y \notin [P_x]$ and $z \in [P_x]$ and if H is the two dimensional variety containing $\frac{x}{2}$, y and z , then either $H \cap P_x = R(\theta, \frac{x}{2})$ or $H \cap P_x$ is a closed segment on $L(z, \frac{x}{2})$ with midpoint $\frac{x}{2}$.

Proof: Since $y \notin [P_x]$, $L(y, \frac{x}{2})$ and $L(z, \frac{x}{2})$ intersect at $\frac{x}{2}$ but are not coincident, hence determine the two dimensional variety H . Either $\theta \in H$ (in which case H is a subspace and clearly contains $R(\theta, \frac{x}{2})$) or $\theta \notin H$ and $L(z, \frac{x}{2}) \cap P_x$ contains a closed segment s with midpoint $\frac{x}{2}$ (by Lemma 3.4). Thus if $\theta \in H$, $R(\theta, \frac{x}{2}) \subset H \cap P_x$ and if $\theta \notin H$, $s \subset H \cap P_x$. Suppose, in the first case that $\theta \in H$, yet that there exists a $u \in H \cap P_x$ such that $u \notin R(\theta, \frac{x}{2})$. Let x_1 and x_2 be distinct points on $R(\theta, \frac{x}{2})$ and consider the convex hull Δ of the set $\{u, x, x_2\}$. (That is Δ is the smallest convex set containing these three points.) Δ has an interior relative to H since if $w \in \Delta$ and w is not on any of the three segments $[x_1, x_2]$, $[u, x_1]$, $[u, x_2]$, there exists a convex open set N containing w and not intersecting any of the three segments, so $N \cap H \subset \Delta$. Then y (as an element of H) is contained in $[\Delta] \subset [P_x]$ contrary to the assumption. Thus, $H \cap P_x \subset R(\theta, \frac{x}{2})$ and, therefore, $H \cap P_x = R(\theta, \frac{x}{2})$. The verification that $H \cap P_x = s$ when $\theta \notin H$ is achieved by replacing $R(\theta, \frac{x}{2})$ by s throughout the preceding argument.

3.20 Definition. The ray $R(\theta, x)$ is an extreme ray of

the cone K if, whenever $[u, v] \subset K$ and $(u, v) \cap R(\theta, x) \neq \emptyset$ it necessarily follows that $[u, v] \subset R(\theta, x)$.

3.21 Lemma. If $R(\theta, x)$ is an extreme ray of K then

$$(a) P_x = [\theta, x]$$

$$(b) [P_x] = L(\theta, x)$$

Proof: Certainly $[\theta, x] \subset K \cap (x - K) = P_x$. Let y be any element of P_x , $y \neq \theta$. Then $x = \frac{1}{2}(2y) + \frac{1}{2}(2(x - y))$ and since $2y$ and $2(x - y)$ belong to K , x is the midpoint of a segment in K . Because $R(\theta, x)$ is an extreme ray, $2y \in R(\theta, x)$ and $2(x - y) \in R(\theta, x)$. Thus, $y \in R(\theta, x)$ and $x - y \in R(\theta, x)$. It follows that $y \in R(\theta, x) \cap (x - R(\theta, x))$. Since $R(\theta, x) = \{z : z = \beta x, \beta \geq 0\}$ and $x - R(\theta, x) = \{z : z = x - \beta x = (1 - \beta)x, \beta \geq 0\} = \{z : z = \alpha x, \alpha \leq 1\}$, $R(\theta, x) \cap (x - R(\theta, x)) = \{z : z = \gamma x, 0 \leq \gamma \leq 1\} = [\theta, x]$. Since y was any element of P_x , $P_x \subset [\theta, x]$ and (a) is true. (b) follows trivially from (a) since the linear extension of the segment $[\theta, x]$ is the line $L(\theta, x)$.

3.22 Theorem. If $x \in K_q$ and $R(\theta, x)$ is an extreme ray of K then the line $L(\theta, x)$ is dense in X .

Proof: Follows immediately from Lemma 3.21 (b).

3.23 An example to show that the situation described in Theorem 3.22 can occur. Let $X = R^2$ with the topology having as a base the family of all open vertical strips of the form $\{(x_1, x_2) : \alpha < x_1 < \beta\}$. It is easy to verify that in this

topology R^2 is locally convex linear topological space with pseudo-norm $p(x_1, x_2) = |x_1|$ determining its topology but is not a normable space since it fails to satisfy the T_1 -separation axiom. It is also easy to see that the x_1 -axis is dense in the space. Let K denote the set of all points (x_1, x_2) with $x_1 \geq 0$ and $x_2 \geq 0$. The non-negative x_1 -axis is an extreme ray of K and the point $(1, 0)$ on this extreme ray is a QI-point of K . We note that the cone K here is not closed. For further details see the discussion of this example in [3].

3.24 Theorem. Suppose K is closed and $K \cup (-K) \neq X$. Then, if $R(\theta, x)$ is an extreme ray of K , $R(\theta, x) \cap K_q = \emptyset$ - i.e. $R(\theta, x) \subset K_p$.

Proof: By Lemma 3.21 (b), $[P_x] = L(\theta, x) = R(\theta, x) \cup R(\theta, -x) \subset K \cup (-K)$. If $x \in K_q$, by Theorem 3.22, $X = \overline{L(\theta, x)} \subset \overline{K \cup (-K)} = \overline{K} \cup \overline{(-K)} = K \cup (-K)$. Since obviously $K \cup (-K) \subset X$ we have that $K \cup (-K) = X$ contradicting our assumption.

The majority of the preceding results demonstrate properties which quasi-interior points have in common with interior points. The following examples are included to indicate some of the differences between these concepts.

3.25 An example of a cone K such that $K^o = K_q = \emptyset$.

Let E be any uncountable set. Let S denote the σ -ring of all

subsets of E . For each finite set F in E let $m(F)$ denote the number of points in F and for any subset A of E with more than a finite number of points let $m(A) = \infty$. (E, S, m) is a measure space. Let X be the space $L(E, S, m)$ of all measurable functions f on E such that $|f|$ is integrable over E . Assign to each f in X the norm $\|f\| = \int_E |f|$. X is a non-separable Banach space. Let K be the class of all functions f in X such that $f(x) \geq 0$ for all x in E . K is a closed cone in X such that $K - K = X$, $K \cap (-K) = \{0\}$, $K^\circ = K_q = \emptyset$. The verifications of these statements together with other properties of K and X may be found in [3].

For another example see [3, example 8.2].

3.26 An example of a cone K such that $K^\circ = \emptyset$ and $K_q \neq \emptyset$.

Let X be the space $L(E, M, m)$ where E is a set, M is the family of m -measurable subsets of E , and with $m(E) < \infty$. It was shown in [2] that if X is not finite dimensional then the cone $K = \{f : f \in X, f(t) \geq 0 \text{ for } t \in E\}$ has no interior. It is easily seen that the function f defined by $f(t) = 1$ for all $t \in E$ is a QI-point of K since the simple functions are dense in L and since if $g(t) = \sum_{i=1}^n \alpha_i \chi_{E_i}(t)$ is any simple function in L (where $\{E_i\}$ is any finite family of disjoint sets in M and χ_{E_i} is the characteristic function of E_i) then if $\gamma = \max_i |\alpha_i|$ and $h(t) = \frac{1}{\gamma} g(t)$, $f \in h + K$ so $h \in [P_f]$ and, hence, $g \in [P_f]$. Since all simple functions thus belong to $[P_f]$, $[P_f]$ is dense in X and $f \in K_q$.

Note that Example 3.8 also shows a cone K with $K^\circ = \emptyset$ yet $K_q \neq \emptyset$. For still another example see [3, example 8.5].

3.27 An example of a line through a QI-point x of a cone K which intersects K only at x . Let $X = c_0$, the space of all real sequences converging to 0. If $x = \{\xi_i\} \in X$ then $\|x\| = \sup_i |\xi_i|$. Let $K = \{x = \{\xi_i\} : x \in X, \xi_i \geq 0 \text{ for all } i\}$. Define $x = \{1, \frac{1}{2}, \frac{1}{4}, \dots, \frac{1}{2^n}, \dots\}$ and $y = \{-1, \frac{1}{2}, -\frac{1}{3}, \dots, \frac{(-1)^n}{n}, \dots\}$. It is easily seen that $x \in K_q$ and $y \notin K$. Consider the line $L(x, y)$. For any $\alpha > 0$ $\alpha y + (1 - \alpha)x = \{(-1)^n \frac{\alpha}{n} + (1 - \alpha) \frac{1}{2^n}\}$. For n odd, $-\frac{\alpha}{n} + \frac{1 - \alpha}{2^n} < -\frac{\alpha}{n} + \frac{1}{2^n} = \frac{-\alpha 2^n + n}{n 2^n}$. Choosing n so that $\alpha > \frac{n}{2^n}$ it follows that $-\alpha 2^n + n < 0$. Thus, for any $\alpha > 0$, $\alpha y + (1 - \alpha)x \notin K_q$ and the only point of the ray $R(x, y)$ in K_q is x itself. Similarly, considering even n with $\alpha < 0$, no point $\alpha y + (1 - \alpha)x$ lies in K_q . Thus $L(x, y)$ intersects K_q only at x .

3.28 An example of a line L through a point $y \notin K$ and a point $x \in K_q$ such that the first point of K on L from y through x is x itself and $L \sim R(x, y) \subset K_q$. As in example 3.27, let $X = c_0$ and K be the positive cone. K_q is the set of elements with strictly positive coordinates. Let $x = \{1, \frac{1}{2}, \frac{1}{4}, \dots, \frac{1}{2^n}, \dots\}$, $y = \{-1, -\frac{1}{2}, -\frac{1}{3}, \dots, -\frac{1}{n}, \dots\}$. Consider $\alpha x + (1 - \alpha)y = \{\frac{\alpha}{2^n} - \frac{(1 - \alpha)}{n}\} = \{\frac{n\alpha - (1 - \alpha)2^n}{n 2^n}\}$. If, for some α with $\alpha < 1$, $\alpha x + (1 - \alpha)y \in K_q$, then $n\alpha - (1 - \alpha)2^n > 0$ for all n . This implies that $\frac{\alpha}{1 - \alpha} > \frac{2^n}{n}$.

for all n . But $\frac{2^n}{n}$ can be made arbitrarily large. Thus, the ray $R(x, y)$ contains no points of K_q other than x . If $\alpha > 1$, $\alpha x + (1 - \alpha)y = \{\frac{n\alpha + (\alpha - 1)2^n}{n2^n}\}$ and each coordinate is positive. Thus $L \cap R(x, y)$ lies entirely in K_q .

3.29 An example of a line L which intersects K in a segment one end of which is a QI-point and the other a QB-point. Let X , K and x be as in examples 3.27 and 3.28. Let $y = \{\frac{1}{4}, \frac{1}{8}, \frac{1}{12}, \dots, \frac{1}{4n}, \dots\}$. Then $\alpha x + (1 - \alpha)y = \{\frac{\alpha}{2^n} + \frac{(1 - \alpha)}{4n}\} = \{\frac{4\alpha n + (1 - \alpha)2^n}{4n2^n}\}$. If $\alpha = -1$, $4\alpha n + (1 - \alpha)2^n = -4n + 2^{n+1} = 2(2^n - 2n) = 0$ if $n = 1, 2$ and > 0 if $n > 2$. Thus, the line $L(x, y)$ contains the QB-point $-x + 2y$. If $-1 < \alpha \leq 1$, $4\alpha n + (1 - \alpha)2^n > 0$ for all n . Thus the segment $(-x + 2y, x] \subset K_q$. If $\alpha > 1$ then, since $\frac{n}{2^n} \rightarrow 0$, $4\alpha \frac{n}{2^n} + (1 - \alpha) < 0$ for a sufficiently large n hence for large enough n , $\frac{\alpha}{2^n} + \frac{(1 - \alpha)}{4n} < 0$ and $\alpha x + (1 - \alpha)y \notin K$. Thus, $L(x, y) \cap K_q = (-x + 2y, x]$. Finally, if $\alpha < -1$, then for $n = 1$, $\frac{\alpha}{2} + \frac{(1 - \alpha)}{4} = \frac{1 + \alpha}{4} < 0$ so $\alpha x + (1 - \alpha)y \notin K$. Thus, $L(x, y) \cap K = [-x + 2y, x]$.

3.30 An example of a line L which intersects a cone K in a segment with endpoints both of which are QI-points. Let $X = L[0, 1]$, the Banach space of all Lebesgue measurable functions absolutely integrable over the interval $[0, 1]$. Let $K = \{f : f \in X, f(t) \geq 0 \text{ a.e.}\}$. Then $K_q = \{f : f \in X, f(t) > 0 \text{ a.e.}\}$. Let $f_1(t) = t^{1/2}$ and $f_2(t) = (1 - t)^{1/2}$. Certainly $f_1, f_2 \in K_q$. Consider $L(f_1, f_2) =$

$\{\alpha f_1 + (1 - \alpha)f_2 : -\infty < \alpha < +\infty\}$. Clearly $[f_1, f_2] \subset K$. If $\alpha \notin [0, 1]$ then $\alpha f_1 + (1 - \alpha)f_2 \notin K$. To see this note that if, for example, $\alpha < 0$ then $g(t) = \alpha f_1(t) + (1 - \alpha)f_2(t) = \alpha t^{1/2} + (1 - \alpha)(1 - t)^{1/2} < 0$ whenever $((1 - t)/t)^{1/2} > (\alpha - 1)/\alpha$ - i.e. for all $t < \alpha^2/(2\alpha^2 - 2\alpha + 1)$. Thus $L(f_1, f_2) \cap K = [f_1, f_2]$.

Only cones with vertex at θ have been considered. Since any convex cone with vertex u in X is the translate $u + K$ of a convex cone K with vertex θ , a natural extension to any convex cone of the quasi-interior point concept can be made.

3.31 Definition. If K is a convex cone (with vertex θ) define the set $(y + K)_q$ to be the set of all points $x \in y + K$ such that $[(y + K) \cap (x - K)]$ is dense in X . $(y + K)_q$ is called the quasi-interior of $y + K$ and each $x \in (y + K)_q$ is a QI-point of $y + K$. (Note that this definition contains definition 3.1 as the special case when $y = \theta$.)

3.32 Theorem. $(y + K)_q = y + K_q$.

Proof: Let z be any element in $y + K_q$. Then $z = y + u$, $u = z - y \in K_q$. Thus $[P_u]$ is dense in X . However, $(y + K) \cap (z - K) = (y + K) \cap (y + u - K) = y + (K \cap (u - K)) = y + P_u$ and, since $[P_u]$ is dense in X , so is $y + P_u$. Thus $[(y + K) \cap (z - K)]$ is dense in X , $z \in (y + K)_q$ and $y + K_q \subset (y + K)_q$. To prove the inclusion in the opposite

direction let z be any element of $(y+K)_q$. Then $[(y+K) \cap (z-K)]$ is dense in X . If $u = z - y$ then $P_u = K \cap (x - K) = -y + ((y+K) \cap (z-K))$ and $[P_u]$ is dense in X . Thus $u = z - y \in K_q$, $z \in y + K_q$ and $(y+K)_q \subset y + K_q$.

4. Support and separation properties.

A hyperplane in X is a maximal proper closed linear subspace. It is easily shown that a necessary and sufficient condition that a set H be a hyperplane is that it be the null space of a non-trivial continuous linear functional on X . The hyperplane $H = f^{-1}[0]$ is said to separate two sets A and B if $x \in A$ implies $f(x) \geq 0$ while $x \in B$ implies $f(x) \leq 0$. $H = f^{-1}[0]$ strictly separates A and B if $f(x) > 0$ for all $x \in A$ while $f(x) < 0$ for all $x \in B$. A hyperplane $H = f^{-1}[0]$ supports a set A if $f(x) \geq 0$ (or $f(x) \leq 0$) for all $x \in A$ and $H \cap A \neq \emptyset$. $x \in A$ is called a support point of A if there is a hyperplane H containing x and supporting A .

The well known Hahn-Banach theorem in its geometric form (c.f. [1]) states that if A is a non-empty open convex set and M is a variety not meeting A then there exists a hyperplane H containing M and not meeting A . In particular, if K is a convex cone with $K^\circ \neq \emptyset$ then every point $x \in K \sim K^\circ$ is a support point of K . Furthermore, if H supports K then $H \cap K^\circ = \emptyset$, and, consequently, every interior point of K is a non-support point of K . The fact that if $K_q \neq \emptyset$ then every point of K_q is a non-support point of K is demonstrated in

Theorem 4.2 below. However, as an example on page 136 of [10] shows, there can be points in K_b which are also non-support points. Some rather restrictive conditions on X which are sufficient to insure that the set of all non-support points of K coincide with K_q are also given in [10]. The results beginning with Theorem 4.4 are concerned with the problem of supporting a cone with non-void quasi-interior by a proper linear subspace, not necessarily a hyperplane, but maximal in a certain well-defined sense.

4.1 Theorem. Let K and K' be cones with non-void quasi-interiors such that $K_q \cap K'_q = \emptyset$. Then if H is a hyperplane separating K and K' , H strictly separates K_q and K'_q .

Proof: Suppose $H = f^{-1}[0]$ separates K and K' . Suppose $x \in H \cap K$. Then, $f(x) = 0$. If y is any element of P_x , $x - y$ is also an element of P_x and $f(x - y) = -f(y)$. If $y \notin H$ then $f(y) \neq 0$ and $f(x - y)$, $f(y)$ are opposite in sign. Both $x - y$ and y belong to K and, because $f(x - y)$, $f(y)$ are opposite in sign, lie on opposite sides of H contradicting that H separates K and K' . Thus $x \in H \cap K$ implies $P_x \subset H$. But then $[P_x] \subset H$ so, since H is a hyperplane, $[P_x]$ is not dense in X . Thus $H \cap K_q = \emptyset$ and assuming that $f(x) \geq 0$ for all $x \in K$, $f(x) > 0$ for all $x \in K_q$. Similarly $f(x) < 0$ for all $x \in K'_q$ and H strictly separates K_q and K'_q .

4.2 Theorem. If H is a supporting hyperplane for K then $H \cap K_q = \emptyset$ - i.e. each QI-point is a non-support point.

Proof: Let H be a supporting hyperplane for K . Then $H = f^{-1}[0]$ and $f(x) \geq 0$ for all $x \in K$. Certainly $\theta \in H$. If $H \cap K = \{\theta\}$ certainly $H \cap K_q = \emptyset$. If $x \in H \cap K$, $x \neq \theta$ then, by an argument analagous to that used in Theorem 4.1, $P_x \subset H$ which implies that $x \notin K_q$. Thus $H \cap K_q = \emptyset$.

4.3 Theorem. Suppose $K_q \neq \emptyset$ and C is a convex set with non-void interior C° such that $C^\circ \cap K_q = \emptyset$. There exists a hyperplane H separating C and K and strictly separating C° and K_q .

Proof: Since C° and K_q are disjoint convex sets, by [6, (8.8)] there exist complementary convex subsets A_1 and A_2 ; $A_1 \cap A_2 = \emptyset$, $A_1 \cup A_2 = X$ with $C^\circ \subset A_1$, and $K_q \subset A_2$. Since C° is open in X it follows from [6, (9.1)] that $\overline{A_1} \cap \overline{A_2}$ is a hyperplane H separating C° and K_q . Furthermore, this same theorem shows that $C^\circ \cap H = \emptyset$ and, by the proof of theorem 4.2, $K_q \cap H = \emptyset$. Thus H strictly separates C° and K_q . $C \cap K$ is a convex set. Assert that $C \cap K \subset H$. If $C \cap K = \emptyset$ this is trivial. If $C \cap K \neq \emptyset$ there exists an $x \in C \cap K$. Assume without loss of generality that $H = f^{-1}[0]$ with $f(K_q) > 0$. First assume that $f(x) > 0$ -i.e. $x \notin H$ and x is on the same side of H as K_q . Since H is closed there is a convex neighborhood N about x such that $f(N) > 0$. Since $x \in C$, $N \cap C^\circ \neq \emptyset$. But this implies that C° contains points on the same side of H as K_q contrary to the fact that H strictly separates C° and K_q . Secondly assume that $y \in C \cap K$ and $f(y) < 0$ -i.e. y and

C° lie on the same side of H . Again there is a convex neighborhood N of y such that $f(N) < 0$. By theorem 3.16, since $K_q \neq \emptyset$, K_q is dense in K so, since $y \in K$ there exist points of K_q in N contrary to the strict separation of C° and K_q . Thus, $C \cap K \subset H$ and H separates C and K .

4.4 Theorem. Suppose $K \cap (-K) = \{\theta\}$, $K_q \neq \emptyset$ and $x_0 \in K_b$, $x_0 \neq \theta$. Then there exists a subset H of X maximal with respect to the properties:

- (i) H is a proper linear subspace of X ,
- (ii) $x_0 \in H$,
- (iii) $H \cap K_q = \emptyset$.

Proof: Note first that K does not consist of a single ray from θ for if it did we would have $K = \{\lambda x_0 : \lambda \geq 0\}$. Since $K_q \neq \emptyset$ there exists a $y \in K_q$, $y \neq \theta$ hence $y = \alpha x_0$, $\alpha > 0$. But then $x_0 = \frac{1}{\alpha} y \in K_q$ contrary to the assumption that $x_0 \in K_b = K \setminus K_q$. Suppose $K = R(\theta, x_0) \cup R(\theta, -x_0) = L(\theta, x_0)$. Then $-K = L(\theta, x_0)$ and $K \cap (-K) = K$ contrary to the assumption that $K \cap (-K) = \{\theta\}$. Thus, K contains at least two independent rays from θ and X contains at least two distinct lines through θ -i.e. X has dimension at least 2. Consider the line $L(\theta, x_0)$ and the ray $R(\theta, x_0)$. Clearly $L(\theta, x_0) \cap K = R(\theta, x_0)$ and $L(\theta, x_0) \cap K_q = (R(\theta, x_0) \cap K_q) \cup (R(\theta, -x_0) \cap K_q) = \emptyset$. Thus $L(\theta, x_0)$ is a proper linear subspace of X containing x_0 with $L(\theta, x_0) \cap K_q = \emptyset$. That is $L(\theta, x_0)$ satisfies (i), (ii) and (iii), and the family \mathcal{S} of all subsets of X satisfying

(i), (ii) and (iii) is non-void. Let \mathcal{J} be partially ordered by inclusion and let $\mathcal{L} = \{H_\alpha\}$ be a maximal linearly ordered subfamily of \mathcal{J} . Let $H = \bigcup_{\alpha} H_{\alpha}$. Clearly H satisfies (i), (ii) and (iii) and is maximal with respect to satisfying these properties.

4.5 Theorem. Let K and H be as in Theorem 4.4.

Define $S_1 = H + K_q$ and $S_2 = H - K_q$. Then

$$(I) \quad H \cap S_1 = H \cap S_2 = S_1 \cap S_2 = \emptyset,$$

$$(II) \quad H \cup S_1 \cup S_2 = X,$$

$$(III) \quad K \subset H \cup S_1$$

$$(IV) \quad K_q \subset S_1$$

$$(V) \quad S_1 \text{ and } S_2 \text{ are symmetric with respect to } H \text{ in}$$

the sense that if $x_1 \in S_1$ and $y \in H$ then there exists an $x_2 \in S_2$ with $y = \frac{1}{2}x_1 + \frac{1}{2}x_2$.

Proof: If $y \in H \cap S_1 = H \cap (H + K_q)$ then $y = u + z \in H$ with $z \in K_q$ so $z = y - u \in K_q \cap H$ contradicting Theorem 4.4 (iii). Thus $H \cap S_1 = \emptyset$. If $y \in H \cap S_2 = H \cap (H - K_q)$ then $y = u - z \in H$ with $z \in K_q$ so $z = u - y \in K_q \cap H$, again contradicting Theorem 4.4 (iii). Thus $H \cap S_2 = \emptyset$. Suppose $y \in S_1 \cap S_2$. Since $y \in S_1 = H + K_q$, $y = u + z$ where $u \in H$, $z \in K_q$ and since $y \in S_2 = H - K_q$, $y = v - w$ where $v \in H$, $w \in K_q$. Thus $u + z = v - w$ so $u = v - (w + z)$. But $v \in H$ and $w, z \in K_q$ imply $u \in H - K_q = S_2$. Thus $u \in H \cap S_2$ contradicting that $H \cap S_2 = \emptyset$. Thus (I) has been proved.

Suppose, contrary to (II) that there exists a $y \in X$ such

that $y \notin H \cup S_1 \cup S_2$. Let $H' = H + Ry = \{u + \alpha y : u \in H, \alpha \in R\}$. Then H' is a linear subspace of X properly containing H and, because H is maximal with respect to (i), (ii) and (iii) of Theorem 4.4, $H' \cap K_q \neq \emptyset$. Let $v \in H' \cap K_q$. Then $v = \alpha y + w$ where $\alpha \neq 0$ and $w \in H$. If $\alpha > 0$ let $v' = \frac{1}{\alpha}v = y + \frac{1}{\alpha}w$. Since $v \in K_q$ so is $v' \in K_q$. Let $z = -\frac{1}{\alpha}w$. Then $z \in H$ and $y = z + v' \in H + K_q = S_1$ contrary to the assumption. If $\alpha < 0$ let $v' = (-\frac{1}{\alpha})v = -y + (-\frac{1}{\alpha})w$. Then, since $v \in K_q$, $v' \in K_q$. Let $z = (-\frac{1}{\alpha})w$. Then $z \in H$ and $y = z - v' \in H - K_q = S_2$, another contradiction. Thus (II) holds.

If $K \not\subset H \cup S_1$ then there exists a $z \in K \cap S_2$. Since $z \in S_2$, $z = u - v$ where $u \in H$ and $v \in K_q$. But then $u = z + v \in K + K_q \subset K_q$ (c.f. Theorem 3.14 (b)) contradicting that $H \cap K_q = \emptyset$. This proves (III).

(IV) is obvious since $K_q = \theta + K_q \subset H + K_q = S_1$.

Let y be any element of H and let x_1 be any element of S_1 . Then $x_1 = u + v$ where $u \in H$ and $v \in K_q$. Define $x_2 = 2y - x_1 = 2y - (u + v) = (2y - u) - v \in H - K_q = S_2$. Clearly $\frac{1}{2}(x_1 + x_2) = y$ so (V) holds and the theorem is proved.

4.6 Theorem. Let K , H and x_0 be as in Theorem 4.4.

If $u \in H$ and $P_u = K \cap (u - K)$ then $P_u \subset H$.

Proof: Suppose, contrary to the desired conclusion, that there exists a point $z \in P_u \sim H$. Then, if $H' = H + Rz$, the maximality of H relative to (i), (ii) and (iii) of Theorem 4.4 implies that $H' \cap K_q \neq \emptyset$. Let $v \in H' \cap K_q$. Then

$v = \alpha z + w$ where $w \in H$, $\alpha \in \mathbb{R}$ and $\alpha \neq 0$ (since $v \notin H$). Since v and z belong to K ($v \in K_q \subset K$ and $z \in P_u \subset K$) we have that the segment $[v, z] \subset K$. If $v' = \beta v + (1 - \beta)z$ with $0 < \beta \leq 1$ and $v'' = \frac{1}{\beta}v' = v + \frac{(1 - \beta)}{\beta}z$ then $v = v'' - \frac{(1 - \beta)}{\beta}z \in K \cap (v'' - K) = P_{v''}$ so $P_v \subset P_{v''}$ and $v'' \in K_q$. But then $v' = \beta v'' \in K_q$ and $[v, z] \subset K_q$. Since P_u is symmetric about $\frac{u}{2}$, $z \in P_u$ implies $u - z \in P_u$. Also $u - z \notin H$ since $u \in H$ and $z \notin H$. Thus $u - z \in P_u \sim H$. Let $H'' = \{t + \alpha(u - z) : t \in H \text{ and } \alpha \in \mathbb{R}\}$. If $w = t + \alpha(u - z) \in H''$ then $w = (t + \alpha u) + (-\alpha)z \in H + \mathbb{R}z = H'$ and $H'' \subset H'$. If $w \in H'$ then $w = u + \alpha z = (1 + \alpha)u + (-\alpha)(u - z) \in H''$ so $H' \subset H''$ and $H' = H''$. Thus, $v \in K_q \cap H''$, $v \notin H$ so $v = \alpha'(u - z) + w'$ with $w' \in H$ and $\alpha' \neq 0$ in \mathbb{R} . Since $u - z, v \in K$ so is $[u - z, v] \subset K$. If $v' = \beta(u - z) + (1 - \beta)v$ where $0 \leq \beta < 1$ then $v'' = \frac{1}{1 - \beta}v' = \frac{\beta}{1 - \beta}(u - z) + v$ and $v = v'' - \frac{\beta}{1 - \beta}(u - z) \in K \cap (v'' - K) = P_{v''}$. Thus v'' and hence v' belong to K_q and $(u - z, v] \subset K_q$. We know that $v = \alpha z + w$ where $w \in H$ and $\alpha \in \mathbb{R}$ with $\alpha \neq 0$. Suppose first that $\alpha > 0$. Then since $u, w \in H$, the point $x = (\frac{1}{1 + \alpha})w + (\frac{\alpha}{1 + \alpha})u$ belongs to H . Also $x = \frac{1}{1 + \alpha}(\alpha z + w) + (\frac{\alpha}{1 + \alpha})(u - z) = \frac{1}{1 + \alpha}v + \frac{\alpha}{(1 + \alpha)}(u - z) \in (u - z, v) \subset K_q$. This contradicts that $H \cap K_q = \emptyset$. The theorem is proved.

4.7 Theorem. Let K, H, S_1 and S_2 be as given in Theorem 4.5.

(a) If $\lambda > 0$ then $\lambda S_1 \subset S_1$ and $\lambda S_2 \subset S_2$.

(b) $S_1 + S_1 \subset S_1, S_2 + S_2 \subset S_2$.

(c) $S_1 \cup \{\theta\}$ and $S_2 \cup \{\theta\}$ are convex cones.

(d) $H \cup S_1$ and $H \cup S_2$ are convex cones; $S_1 \cup S_2$ is not convex.

(e) $-S_2 = S_1$.

(f) $H + S_1 \subset S_1$, $H + S_2 \subset S_2$.

(g) $H - S_1 \subset S_2$, $H - S_2 \subset S_1$.

(h) $S_2 - S_1 \subset S_2$, $S_1 - S_2 \subset S_1$.

Proof:

(a) If $x \in S_1$ then $x = u + w$ where $u \in H$ and $w \in K_q$. Then $\lambda x = \lambda u + \lambda w$. Clearly $\lambda u \in H$ and since for $\lambda > 0$, $\lambda K_q \subset K_q$, $\lambda w \in K_q$. Thus $\lambda > 0$ implies $\lambda x \in H + K_q = S_1$. An analagous argument verifies that $\lambda S_2 \subset S_2$ if $\lambda > 0$.

(b) If $x, y \in S_1$ then $x = u + w$ and $y = s + t$ where $u, s \in H$ and $w, t \in K_q$. Clearly $x + y = (u + s) + (t + w) \in H + K_q = S_1$. Similarly for S_2 .

(c) This follows immediately from (a) and (b).

(d) Let $x \in H \cup S_1$. Then $x \in H$ or $x \in S_1$. If $\lambda \geq 0$ and $x \in H$, $\lambda x \in H$. If $x \in S_1$ and $\lambda > 0$ then $\lambda x \in S_1$. If $x \in S_1$ and $\lambda = 0$, $\lambda x = \theta \in H$. Thus, $x \in H \cup S_1$ and $\lambda \geq 0$ imply $\lambda x \in H \cup S_1$. If $x, y \in H$ then $x + y \in H$. If $x, y \in S_1$ then $x + y \in S_1$. If $x \in H$ and $y \in S_1$ then $y = u + v$ where $u \in H$ and $v \in K_q$ so $x + y = x + (u + v) = (x + u) + v \in S_1$. In any case, if $x, y \in H \cup S_1$ then $x + y \in H \cup S_1$ then $x + y \in H \cup S_1$. Thus, $H \cup S_1$ is a convex cone. Similarly for $H \cup S_2$. To prove that $S_1 \cup S_2$ is not convex we need only exhibit two points $u, v \in S_1 \cup S_2$ such that $[u, v] \cap H \neq \emptyset$.

Let $u \in S_1$ and $x \in H$. Then $v = 2x - u \in S_2$ and $x \in (u, v) \cap H$.

(e) Suppose $x \in -S_2$. Then $-x \in S_2$ and $-x = u - v$ where $u \in H$ and $v \in K_q$ and $x = -u + v \in H + K_q = S_1$. Thus $-S_2 \subset S_1$. Now suppose $x \in S_1$. Then $x = u + v$ with $u \in H$ and $v \in K_q$ so $-x = (-u) - v \in H - K_q = S_2$ and $x \in -S_2$. Thus $S_1 \subset -S_2$ and the equality holds.

(f) If $x = u + v$ where $u \in H$ and $v \in S_1$ then $v = u' + v'$ where $u' \in H$ and $v' \in K_q$ and, $x = u + (u' + v') = (u + u') + v' \in H + K_q = S_1$. Thus, $H + S_1 \subset S_1$. Similarly $H + S_2 \subset S_2$.

(g) This follows trivially from (e) and (f).

(h) This follows trivially from (b) and (f).

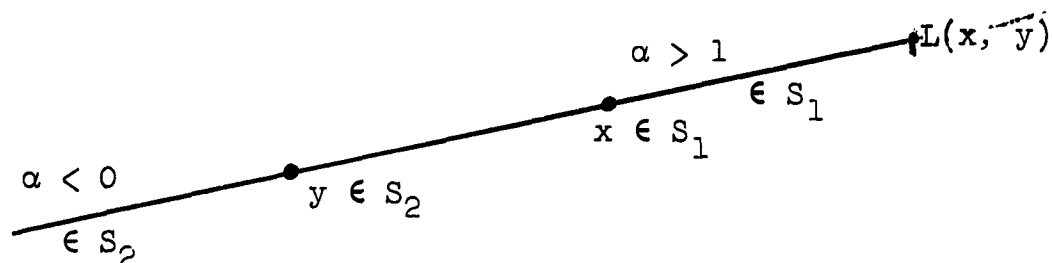
4.8 Theorem. Let K , H , S_1 and S_2 be as given in Theorem 4.5. If $y \in K_q$ and $H_1 = H + Ry$ define $H_1^+ = \{u + \alpha y : u \in H, \alpha > 0\}$ and $H_1^- = \{u + \alpha y : u \in H, \alpha < 0\}$. Then $H_1 = H_1^+ \cup H_1^- \cup H$, $H \cap H_1^+ = H \cap H_1^- = H_1^+ \cap H_1^- = \emptyset$, $H_1 \cap S_1 = H_1^+$, and $H_1 \cap S_2 = H_1^-$.

Proof: These statements follow immediately from the definitions and the fact that $\lambda K_q \subset K_q$ if $\lambda > 0$.

4.9 Theorem. Let K , H , S_1 and S_2 be as given in Theorem 4.5. Suppose $x \in S_1$, $y \in S_2$ and $L(x, y) = \{\alpha x + (1 - \alpha)y : \alpha \in R\}$ is the line through x and y . If $\alpha \leq 0$, $\alpha x + (1 - \alpha)y \in S_2$ and if $\alpha \geq 1$, $\alpha x + (1 - \alpha)y \in S_1$. In particular, if $x \in S_1$ and $[y, x) \subset S_2$ then $L(x, y) \cap S_1 = \{\alpha x + (1 - \alpha)y : \alpha \geq 1\}$, $L(x, y) \cap S_2 = \{\alpha x + (1 - \alpha)y : \alpha < 1\}$.

hence $L(x, y) \subset S_1 \cup S_2$ and $L(x, y) \cap H = \emptyset$.

Proof: Let $z = \alpha x + (1 - \alpha)y$ where $\alpha < 0$. Since $x = u + v$ and $y = u' - v'$ where $u, u' \in H$ and $v, v' \in K_q$,
 $z = \alpha(u + v) + (1 - \alpha)(u' - v') = \alpha u + (1 - \alpha)u' + \alpha v - (1 - \alpha)v'$.
 Since $\alpha < 0$, $1 - \alpha > 0$ and $-(1 - \alpha) < 0$. Thus $\alpha v \in -K_q$ and $-(1 - \alpha)v' \in -K_q$ so $\alpha v - (1 - \alpha)v' \in -K_q$ and $z \in H - K_q = S_2$.
 Let $z = \alpha x + (1 - \alpha)y$ where $\alpha > 1$. Then $z = \alpha(u + v) + (1 - \alpha)(u' - v') = \alpha u + (1 - \alpha)u' + \alpha v - (1 - \alpha)v'$. Since $\alpha > 1$, $1 - \alpha < 0$ so $-(1 - \alpha) > 0$ and $\alpha v, -(1 - \alpha)v' \in K_q$. Thus, $\alpha v - (1 - \alpha)v' \in K_q$ and $z \in H + K_q = S_1$.



4.10 Theorem. Let K, H, S_1 and S_2 be as in Theorem 4.5. H is dense in X if and only if $S_1 \cup S_2$ has a void interior.

Proof: By Theorem 4.5 (I, II), $S_1 \cup S_2 = X \sim H$. Thus, $x \in \overline{H}$ if and only if $x \notin (S_1 \cup S_2)^\circ$.

5. A Conjecture.

Throughout this section let K, H, x_0, S_1 and S_2 be as given in Theorems 4.4 and 4.5. The following statement was given by Fullerton. Because, to this date, the writer has not managed either to prove its validity or to find a

counter-example, it will be labeled here as:

5.1 Conjecture. Either H is a hyperplane or H , S_1 and S_2 are dense in X .

Fullerton left the following sketch of a "proof."
 Suppose $x \in X \sim H$. Let $H_1 = \{u + \alpha x : u \in H, \alpha \in R\} = H + Rx$.
 Then H_1 is necessarily dense in X . If H is closed, clearly H_1 is closed, and being dense in X , $H_1 = X$. Thus H is a closed linear subspace of X of deficiency one and is thus a hyperplane. Suppose H is not dense in X . Then \bar{H} is a proper linear subspace of X and is therefore contained in a hyperplane M . If H is properly contained in M , let $x \in M \sim H$. Then $H_1 \subset M$ and cannot be dense in X contrary to the underlined statement above. Thus $H = M$ and H is closed.

This "proof" is certainly valid if the underlined statement is true. It is the validity of the underlined statement about which there is some question.

In keeping with the writer's philosophy concerning technical reports as outlined in the introduction, the remainder of this section is devoted to the several attempts made, so far unsuccessfully, to prove the conjecture. Certainly little, if any, of the material to follow will appear in the paper which will evolve from this report.

5.2 Lemma. If $x \in X \sim H$ and $H_1 = H + Rx$ then $H_1 \cap K_q \neq \emptyset$. If $v \in H_1 \cap K_q$ then $H_1 = H + Rv$.

Proof: Either $H_1 = X$, in which case, trivially, $H_1 \cap K_q \neq \emptyset$, or H_1 is a proper linear subspace of X . Since H_1 properly contains H and since H is maximal with respect to (i), (ii) and (iii) of Theorem 4.4, $H_1 \cap K_q \neq \emptyset$. If $v \in H_1$ and $v \notin H$ (clearly this is the case if $v \in H_1 \cap K_q$ since $H \cap K_q = \emptyset$) we have $v = u + \alpha x$ with $\alpha \neq 0$ for some $u \in H$. Then $x = \frac{1}{\alpha}(v - u)$ so, if $y \in H_1$, $y = u' + \alpha'x = u' + \alpha'\frac{1}{\alpha}(v - u) = (u' - \frac{\alpha'}{\alpha}u) + \frac{\alpha'}{\alpha}v \in H + Rv$. Conversely, if $z = u' + \beta v \in H + Rv$ then $z = u' + \beta(u + \alpha x) = (u' + \beta u) + \beta \alpha x \in H + Rx = H_1$. Thus $H_1 = H + Rv$.

5.3 Lemma. If $v \in H_1 \cap K_q$ and $s \in P_v$ then either $s \in H_1$ or there exists a $y \in K_q$ such that $s + y \in H_1$.

Proof: If $s \in P_v$ then $s = v - z$ where $s, z \in K$. Now since $K \subset S_1 = H + K_q$, $z = u + y$ where $u \in H$ and $y \in K_q$. Thus $s = v - (u + y)$ so $s + y = -u + v \in H + Rv = H_1$.

5.4 Assumption. If $v \in H_1 \cap K_q$ then $P_v \subset H_1$.

5.5 Proof of Conjecture 5.1 assuming 5.4. If $P_v \subset H_1$ then $[P_v] \subset H_1$. Since $v \in K_q$, $[P_v]$ is dense in X hence H_1 is dense in X . The underlined statement in the paragraph following 5.1 is thus verified and the conjecture is valid.

Thus, if 5.4 were valid the conjecture would be proved.

5.5 Lemma. If $K_q \subset H_1$ then 5.4 holds.

Proof: By 5.3, $s \in P_v$ implies $s \in H_1$ or $s + y \in H_1$ for

some $y \in K_q$. If $K_q \subset H_1$ then $y = u + \alpha v$ with $u \in H$, $\alpha \in R$ and $s + y \in H_1$ implies $s + y = s + (u + \alpha v) = u' + \alpha' v$. This implies $s = (u' - u) + (\alpha' - \alpha)v \in H_1$ so $P_v \subset H_1$.

Thus, if it could be verified that $K_q \subset H_1$, the conjecture would be proved.

Another approach to the problem uses the concept of points linearly accessible from a set. According to Klee [6],

5.6 Definition. A point $y \in X$ is linearly accessible from a set $A \subset X$ if $(y, x] \subset A$ for some $x \in A$, $x \neq y$. The union of A with the set of all points linearly accessible from A will be denoted by lin A. A subset $B \subset X$ is called ubiquitous if $\text{lin } B = X$.

The following results are to be found in [6] and are numbered as they are there.

[6, (8.4)] If C is a convex set with non-empty interior and C is ubiquitous then $C = X$.

[6, (8.9)] If C and D are complementary convex subsets of X and $M = \text{lin } C \cap \text{lin } D$ then either M is a maximal variety or $M = X$.

[6, (7.1)] Each maximal variety is either closed or dense.

Now if we could show that $H = \text{lin } C \cap \text{lin } D$ for complementary convex sets C and D (for example $C = H \cup S_1$, $D = S_2$)

then by [6, (8.9)] H is a maximal variety so by [6, (7.1)], H is either closed or dense.

5.7 Lemma. $H \subset (\text{lin } S_1) \cap (\text{lin } S_2)$.

Proof: Let u be any element of H . Let v_1 be any element of S_1 . Certainly $v_1 \neq u$. Define $v_2 = 2u - v_1$. Then $v_2 \in S_2$ and $v_2 \neq u$. Consider $[v_2, u) = \{\lambda u + (1 - \lambda)v_2 : 0 \leq \lambda < 1\}$. Clearly $[v_2, u) \subset H + S_2 \subset S_2$ (Theorem 4.7 (f)). Also $[v_1, u) = \{\lambda u + (1 - \lambda)v_1 : 0 \leq \lambda < 1\} \subset H + S_1 \subset S_1$. Thus, $u \in (\text{lin } S_1) \cap (\text{lin } S_2)$.

5.8 Lemma. $H \cup S_1 \subset \text{lin } S_1$.

Proof: $H \subset \text{lin } S_1$ by Lemma 5.7 and $S_1 \subset \text{lin } S_1$ by definition 5.6.

5.9 Lemma. $A \subset B$ implies $\text{lin } A \subset \text{lin } B$.

Proof: If $x \in A$ and $A \subset B$ then $x \in B \subset \text{lin } B$. Suppose $x \in \text{lin } A$, $x \notin A$. Then there exists a $y \in A \subset B$ such that $[y, x) \subset A \subset B$ so $x \in \text{lin } B$.

5.10 Lemma. $\text{lin } (H \cup S_1) \subset \text{lin } S_1$.

Proof: Suppose $p \in \text{lin } (H \cup S_1)$. If $p \in H \cup S_1$ then, by Lemma 5.8, $p \in \text{lin } S_1$. If $p \notin H \cup S_1$ then, because $p \in \text{lin } (H \cup S_1)$ there exists a $y \in H \cup S_1$ such that $(p, y] \subset H \cup S_1$. Either $y \in H$ or $y \in S_1$ since $H \cap S_1 = \emptyset$. Suppose $y \in H$. Let $v \in (p, y)$. If $v \in H$ the entire line through v and y is in H because H is a subspace. But this contradicts that $p \notin H \cup S_1$. Thus, $v \in S_1$, $v =$

$\alpha y + (1 - \alpha)p$, $0 < \alpha < 1$. Since $v \in S_1$, $v = u + w$ where $u \in H$ and $w \in K_q$. Thus, $u + w = \alpha y + (1 - \alpha)p$ and $(1 - \alpha)p = (u - \alpha y) + w \in H + K_q = S_1$. But $\lambda S_1 \subset S_1$ for $\lambda > 0$ so, since $\frac{1}{1 - \alpha} > 0$, $p \in S_1$ again contradicting that $p \notin H \cup S_1$. Thus $y \notin H$ and, necessarily, $y \in S_1$. Thus $p \in \text{lin } S_1$ and $\text{lin } (H \cup S_1) \subset \text{lin } S_1$.

5.11 Lemma. $\text{lin } (H \cup S_1) = \text{lin } S_1$.

Proof: Since $S_1 \subset H \cup S_1$, $\text{lin } S_1 \subset \text{lin } (H \cup S_1)$ by Lemma 5.9. The result follows from Lemma 5.10.

5.12 Lemma. Let $M = \text{lin } S_1 \cap \text{lin } S_2$. Then

(a) $H \subset M$,

(b) either $M = X$, M is a hyperplane, or M

is a proper linear subspace dense in X .

Proof: $H \cup S_1$ is convex (Theorem 4.7 (d)), S_2 is convex (Theorem 4.7 (c)), $(H \cup S_1) \cap S_2 = \emptyset$ (Theorem 4.5 (I)), and $(H \cup S_1) \cup S_2 = X$ (Theorem 4.5 (II)). Thus $H \cup S_1$ and S_2 are complementary convex sets. $\text{lin } S_1 = \text{lin } (H \cup S_1)$ by Lemma 5.11. Thus $M = \text{lin } (H \cup S_1) \cap \text{lin } S_2$ is either a maximal variety or $M = X$. That $H \subset M$ is Lemma 5.7. Since H is a linear subspace, $\theta \in H$ so $\theta \in M$ and M is a subspace, hence if $M \neq X$, M is a maximal proper linear subspace. By [6, (7.1)], if $M \neq X$, M is either a hyperplane or M is dense in X .

5.13 Lemma. If $M = X$ (where $M = \text{lin } S_1 \cap \text{lin } S_2$) then

(a) X is infinite dimensional.

$$(b) \quad X = \text{lin } S_1 = \text{lin } S_2 = \text{lin } (H \cup S_1) = \text{lin } (H \cup S_2),$$

$$(c) \quad S_1^{\circ} = S_2^{\circ} = (H \cup S_1)^{\circ} = (H \cup S_2)^{\circ} = \emptyset.$$

Proof: Suppose $M = X = \text{lin } S_1 \cap \text{lin } S_2$. Then $\text{lin } S_1 = X = \text{lin } S_2$. Since $\text{lin } (H \cup S_1) = \text{lin } S_1$ and $\text{lin } (H \cup S_2) = \text{lin } S_2$, (b) follows. Thus $M = X$ implies the sets S_1 , S_2 , $H \cup S_1$ and $H \cup S_2$ are all ubiquitous. S_1 is a convex proper subset of X and is ubiquitous. By a theorem of Klee [6, (8.1)], this can happen if and only if X is infinite dimensional so (a) holds. Finally, by [6, (8.4)], since S_1 , S_2 , $H \cup S_1$ and $H \cup S_2$ are ubiquitous and yet proper subsets of X , they must all have void interiors.

5.14 Lemma. If $M = X$ (where $M = \text{lin } S_1 \cap \text{lin } S_2$) then H is not a hyperplane.

Proof: Assume H is a hyperplane. Then H is not dense in X so, by Theorem 4.10, $S_1 \cup S_2$ has a non-void interior. Because $M = X$, from Lemma 5.13 (c), $S_1^{\circ} = S_2^{\circ} = \emptyset$. This, together with the fact that S_1 and S_2 are convex give that $S_1 \cup S_2$ is polygonally connected. But this implies that $S_1 \cup S_2 = X \sim H$ is connected -i.e. H does not separate X . By a theorem of Klee, [7, (2.1)], H is not a hyperplane.

5.15 Lemma. Suppose $H \neq M$ (where $M = \text{lin } S_1 \cap \text{lin } S_2$). Then there exists a line L in X such that $L \cap S_1 \neq \emptyset$, $L \cap S_2 \neq \emptyset$ and $L \cap H = \emptyset$ (i.e. $L \subset S_1 \cup S_2$).

Proof: If $H \neq M$, since $H \subset M$ (Lemma 5.12 (a)) there

exists an $x \in M$, $x \notin H$. Since $x \notin H$ and $X = H \cup S_1 \cup S_2$, $x \in S_1$ or $x \in S_2$. Assume without loss of generality that $x \in S_1$. Since $x \in M = \text{lin } S_1 \cap \text{lin } S_2$, $x \in S_1 \cap \text{lin } S_2$. Since $x \in \text{lin } S_2$ and $x \notin S_2$ ($S_1 \cap S_2 = \emptyset$) there exists a $y \in S_2$ such that $[y, x) \subset S_2$. It follows immediately from Theorem 4.9 that $L(x, y) \subset S_1 \cup S_2$.

5.16 Lemma. If H is dense in X so are S_1 and S_2 .

Proof: Let x be any element of H . Then $x \in H \cup S_1 \subset \text{lin } (H \cup S_1) = \text{lin } S_1 \subset \overline{S_1}$ so $H \subset \overline{S_1}$. Thus, $\overline{H} = X$ implies $\overline{S_1} = X$. Similarly for S_2 .

5.17 Lemma. The following statements are equivalent.

- (a) $M \cap K_q = \emptyset$ (where $M = \text{lin } S_1 \cap \text{lin } S_2$).
- (b) $K_q \cap \text{lin } S_2 = \emptyset$.
- (c) $S_1 \cap \text{lin } S_2 = \emptyset$.

Furthermore, if $x \in S_1 \cap \text{lin } S_2$ then $x = u + v$ where $u \in H$ and $v \in K_q \cap \text{lin } S_2$.

Proof: Suppose $x \in S_1 \cap \text{lin } S_2$. Then, since $x \in S_1$, $x = u + v$ where $u \in H$ and $v \in K_q$. Since $x \in \text{lin } S_2$ and $x \notin S_2$ there exists a $y \in S_2$ such that $[y, x) \subset S_2$. Thus, for any α , $0 \leq \alpha < 1$, $\alpha x + (1 - \alpha)y = \alpha(u + v) + (1 - \alpha)y = u_\alpha + v_\alpha$ where $u_\alpha \in H$ and $v_\alpha \in K_q$. It follows that $\alpha v + (1 - \alpha)y = (u_\alpha - \alpha u) + v_\alpha \in S_2$ and $[y, v) \subset S_2$. Thus $v \in K_q \cap \text{lin } S_2$. Then $K_q \cap \text{lin } S_2 = \emptyset$ implies $S_1 \cap \text{lin } S_2 = \emptyset$. Since $K_q \subset S_1$, clearly $S_1 \cap \text{lin } S_2 = \emptyset$ implies $K_q \cap \text{lin } S_2 = \emptyset$, and (b) and (c) are equivalent. $M \cap K_q =$

$(\text{lin } S_1 \cap \text{lin } S_2) \cap K_q = (K_q \cap \text{lin } S_1) \cap \text{lin } S_2$ so (a) and (b) are equivalent.

A third possible approach to the problem could make use of the following Lemma whose proof can be found in [5, p. 19].

5.18 Lemma. A cone P such that $P \neq \emptyset$, $P \neq X$ is a half-space if and only if it has a non-void directional-interior, P_d and if the union of the sets P and $-P_d$ is the entire space. If P is a half-space $P \cap (-P)$ is a maximal linear subspace.

In order to make use of Lemma 5.18, set $H \cup S_1 = P$. Note that $H \cup S_1$ is a convex cone according to Theorem 4.7 (d). We ask the following questions. 1) Is P_d non-empty? 2) Is $S_2 = -P_d$? If the answers to both 1) and 2) are yes then, because $(H \cup S_1) \cup S_2 = X$ we have $P = H \cup S_1$ is a half-space. Since $-H = H$ and $-S_1 = S_2$ we have $-P = H \cup S_2$ so $P \cap (-P) = (H \cup S_1) \cap (H \cup S_2) = H$ and therefore H is a maximal linear subspace so is either dense or a hyperplane.

5.19 Lemma. Let $P = H \cup S_1$.

(a) $P_d \subset S_1$.

(b) If $K_q \cap \text{lin } S_2 = \emptyset$ then $S_1 \subset P_d$.

Proof: If $P_d = \emptyset$ trivially $P_d \subset S_1$. Thus suppose $P_d \neq \emptyset$. Let x be any element of P_d . Then, given any $y \in X$ there exists a $z \in (x, y) \cap P$ such that $[x, z] \subset P$. If $y \in S_2$ then $y = u - v$ where $u \in H$ and $v \in K_q$. For some

α , $0 < \alpha < 1$, $z = \alpha y + (1 - \alpha)x$. Since $P_d \subset P = H \cup S_1$ either $x \in H$ or $x \in S_1$. If $x \in H$ then $z = \alpha(u - v) + (1 - \alpha)x = (\alpha u + (1 - \alpha)x) - \alpha v \in H - K_q = S_2$ contrary to the fact that $z \in P$ and $P \cap S_2 = \emptyset$. Thus, $x \in S_1$. Since x was any element of P_d , $P_d \subset S_1$ and (a) is proved.

To prove (b) suppose that $K_q \cap \text{lin } S_2 = \emptyset$. Then by Lemma 5.17, $S_1 \cap \text{lin } S_2 = \emptyset$. This means that if $x \in S_1$ and $y \in S_2$ then $[y, x) \not\subset S_2$. Suppose that x is any element of S_1 . Let $y \in X$, $y \neq x$. If $y \in P = H \cup S_1$ then $[y, x] \subset P$ since, by Theorem 4.7 (d), P is convex. Then, clearly, there exists a $z \in (x, y) \cap P$ such that $[x, z] \subset P$. If $y \notin P$ then, because $X = P \cup S_2$, $y \in S_2$. But $[y, x) \not\subset S_2$ so there exists a $z \in (y, x)$ with $z \notin S_2$ hence $z \in P$. Since P is convex and $x, z \in P$, $[x, z] \subset P$. Thus, $x \in P_d$ and because x was any element of S_1 , $S_1 \subset P_d$.

5.20 Corollary. If $K_q \cap \text{lin } S_2 = \emptyset$ then H is a maximal linear subspace (and is therefore either a hyperplane or dense).

Proof: We saw in Lemma 5.19 that if $K_q \cap \text{lin } S_2 = \emptyset$ then $P_d = S_1 \neq \emptyset$. Since $S_1 = -S_2$ (Theorem 4.7 (e)) the remarks preceding Lemma 5.19 give the conclusion here.

5.21 Theorem. A necessary and sufficient condition that H be maximal linear subspace is that $K_q \cap \text{lin } S_2 = \emptyset$.

Proof: The sufficiency was stated in Corollary 5.20.

Suppose that H is a maximal linear subspace. Then $P = H \cup S_1$ is a half-space and by Lemma 5.18, $P_d \neq \emptyset$ and $X = P \cup (-P_d)$. By Lemma 5.19, $P_d \subset S_1$ so $-P_d \subset -S_1 = S_2$. Since $X = P \cup (-P_d) = P \cup S_2$ and $S_2 \cap P = \emptyset$, $-P_d = S_2$ and $P_d = S_1$. Since $P_d = S_1$ if x is any point in S_1 and y is any point in S_2 then there exists a $z \in (y, x) \cap P$ such that $[z, x] \subset P$. Thus there exists no point $y \in S_2$ such that $[y, x) \subset S_2$ and, consequently, $S_1 \cap \text{lin } S_2 = \emptyset$. By Lemma 5.17, $S_1 \cap \text{lin } S_2 = \emptyset$ if and only if $K_q \cap \text{lin } S_2 = \emptyset$. The theorem is proved.

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